B.Sc. part → 1

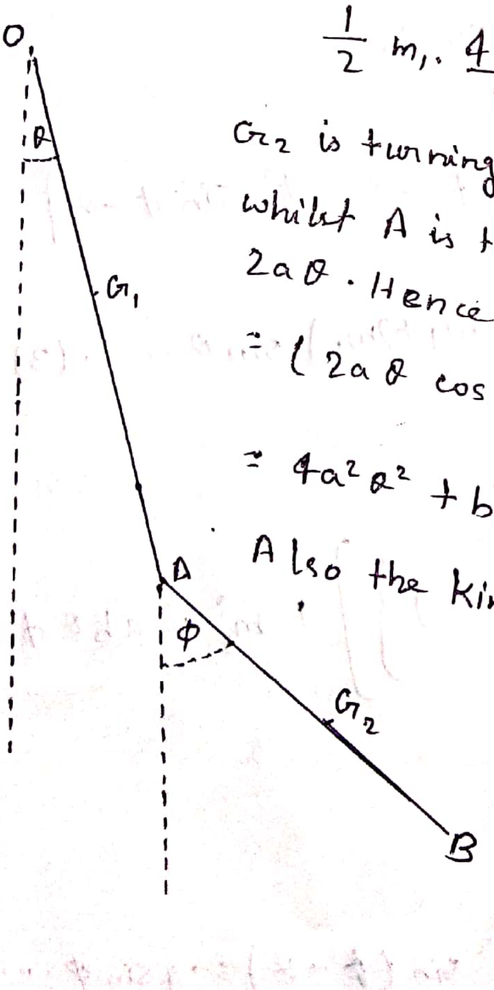
248. When a system is such that coordinates of any particle of it can be expressed in terms of independent coordinates by equations which do not contain differential coefficients with regard to the time, the system is said to be holonomus.

249. Ex. 1. A homogenus rod OA, of mass m_1 and length ~~m2~~ $2a$, is freely hinged at O to a fixed point; at its other is freely attached another homogenous rod AB, of mass m_2 and length $2b$; the system moves under gravity; find equation to determine the motion.

Let G_1 and G_2 be the centres of mass of the rods, and θ and ϕ their inclination to the vertical at time t.

The Kinetic energy of OA is

$$\frac{1}{2} m_1 . \frac{4a^2}{3} . \dot{\theta}^2 .$$



G_2 is turning round A with velocity $b\dot{\phi}$, whilst A is turning round O with velocity $2a\dot{\theta}$. Hence the square of the velocity of G_2

$$= (2a\dot{\theta} \cos\theta + b\dot{\phi} \cos\phi)^2 + (2a\dot{\theta} \sin\theta + b\dot{\phi} \sin\phi)^2$$

$$= 4a^2\dot{\theta}^2 + b^2\dot{\phi}^2 + 4ab\,\dot{\theta}\dot{\phi} \cos(\theta - \phi).$$

Also the Kinetic energy of the rod about G_2

$$= \frac{1}{2} m_2 . \frac{b^2}{3} \dot{\phi}^2 .$$

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$\therefore\ T = \frac{1}{2} m_1 \cdot \frac{4a^2}{3} \dot\theta^2 + \frac{1}{2} m_2 \left[4a^2\dot\theta^2 + b^2\dot\phi^2 + 4ab\dot\theta\dot\phi \cos(\phi-\theta)\right.$

$$+ \frac{1}{2} m_2 \cdot \frac{b^2}{3}\dot\phi^2$$

$$= \frac{1}{2}\left(\frac{m_1}{3} + m_2\right)\cdot 4a^2\dot\theta^2 + \frac{1}{2}m_2 \cdot \frac{4b^2}{3}\dot\phi^2 + \frac{1}{2}m_1 \cdot 4ab\dot\theta\dot\phi \cos(\phi-\theta) \ldots\ldots (1).$$

Also the work function V.

$$= m_1 g a \cos\theta + m_2 g (2a\cos\theta + b\cos\phi) + c \ldots\ldots\ldots (2).$$

Lagrange's θ equation then gives.

$$\frac{d}{dt}\left[\left(\frac{m_1}{3} + m_2\right)\cdot 4a^2\dot\theta + m_2 \cdot 2ab\dot\phi \cos \phi - \theta\right] - 2m_2 ab\dot\theta\dot\phi \sin(\phi-\theta)$$

$$= \frac{d}{dt}\left(\frac{\partial T}{\partial \dot\theta}\right) - \frac{\partial T}{\partial \theta} = \frac{\partial V}{\partial \theta}$$

$$= -(m_1 + 2m_2) g a \sin\theta,$$

i.e $\left(\frac{m_1}{3} + m_2\right) 4a\ddot\theta + 2m_2 b\left[\ddot\phi \cos \phi - \theta - \dot\phi^2 \sin \phi - \theta\right]$

$$= -g(m_1 + 2m_2)\sin\theta \ldots\ldots (3).$$

So the ϕ - equation is

$$\frac{d}{dt}\left[m_2\left\{\frac{4b^2}{3}\dot\phi + 2ab\dot\theta \cos(\phi-\theta)\right\}\right] + m_2 \cdot 2ab\dot\theta\dot\phi \sin(\phi-\theta)$$

$$= m_2 bg \sin\phi,$$

i.e. $\frac{4b}{3}\ddot\phi + 2a\ddot\theta \cos(\phi-\theta) + 2a\dot\theta^2 \sin(\phi-\theta) = -g\sin\phi \ldots\ldots (4).$

Multiplying (3) by $a\dot\theta$, 4 by $m_2 b\dot\phi$, adding and integrating, we have $\frac{1}{2}\left(\frac{m_1}{3}+m_2\right)4a^2\dot\theta^2+\frac{1}{2}m_2 \cancel{b\dot\phi} \cdot 4ab\dot\theta\dot\phi$
$\cos(\phi-\theta) = (m_1+2m_2)ga\cos\theta + m_2 gb\cos\phi + C$ ---------- (5)

This is the equation of Energy.

Again multiplying (3) by a, and (4) by $m_2 b$ we have, on adding,

$$\frac{d}{dt}\left[\left(\frac{m_1}{3}+m_2\right)\cdot 4a^2\dot\theta + m_2\left\{\frac{4b^2}{3}\dot\phi + 2ab(\dot\theta+\dot\phi)\cos(\phi-\theta)\right\}\right] = -ag(m_1+2m_2)\sin\theta - m_2 bg\sin\phi,$$

This is the equation derived by taking moments about O for the system.

Ex. 2. A uniform rod of length $2a$, can turn freely about one ~~end~~, which is fixed. Initially it is inclined at an acute angle α to the downward drawn vertical and it is set rotating about a vertical axis through its fixed end with angular velocity ω. Shew that during the motion, the rod is always inclined to the motion, the rod is always inclined to the vertical at an angle which is $\gtrless \alpha$, according as $\omega^2 \gtrless \frac{3g}{4a\cos\alpha}$, and that in each case its motion is included the inclination α and

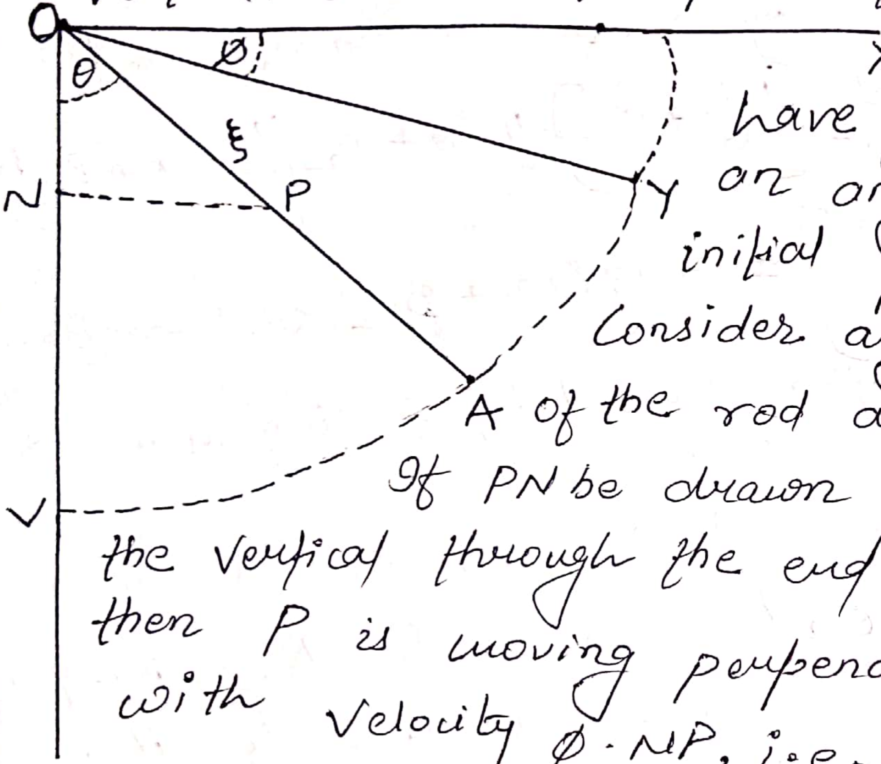
$$\cos^{-1}\left[-n+\sqrt{1-2n\cos\alpha+n^2}\right]$$

$$n = \frac{a\omega^2\sin^2\alpha}{3g}$$

where,

If it be slightly disturbed when revolving steadily at a constant inclination a, shew that the time of a small oscillation is $2\pi\sqrt{\dfrac{4a\cos a}{3g(1+3\cos^2 a)}}$

At any time t let the rod be inclined at θ to the vertical, and let the plane through it and the vertical have turned through an angle ϕ from its initial position.



Consider any element $\frac{d\xi}{2a}\cdot m$ of the rod at P, where $OP = \xi$. If PN be drawn perpendicular to the vertical through the end O of the rod, then P is moving perpendicular to NP with velocity $\dot\phi\cdot NP$, i.e. $\xi\sin\theta\dot\phi$, and it is moving perpendicular to OP in the plane VOA with velocity $\xi\dot\theta$.

$$= \frac{1}{2}\cdot\frac{d\xi}{2a}\cdot m\left[\xi^2\sin^2\theta\dot\phi^2 + \xi^2\dot\theta^2\right].$$

Therefore the whole kinetic energy T

$$= \frac{1}{2}\cdot\frac{m}{2a}\left(\sin^2\theta\dot\phi^2 + \dot\theta^2\right)\int_a^{2a}\xi^2\,d\xi = \frac{2ma^2}{3}\left(\dot\phi^2\sin^2\theta + \dot\theta^2\right)$$

Also the work function V

$$= m\cdot g\cdot a\cos\theta + C.$$

continue